Institute of Research in Mathematical Education

Aix-Marseille University

PLAYING AL - JABR CARDS

1. – The Algebra of Al-Khawarizmi.

A brief historical note : Strictly speaking, **al-jabr** (restoration) corresponds to transforming a subtraction in one member of an equality into an addition in the other member, unlike **al-muqabala** (balancing) which amounts to eliminating the addition of the same number in both members.

These techniques are the basis of **The Compendium of Calculation by Restoration** (al-jabr) **and Comparison** (al-muqabala) (in Arabic: الكتاب المختصر في حساب الجبر والمقابلة, Kitāb al-mukhtaṣar fī ḥisāb al-jabr wa-l-muqābala) which is a historical book of mathematics written between 813 and 833 by the Persian mathematician Al-Khawarizmi. In this book Al-Khawarizmi lays the foundations of algebra by being the first to systematically study the resolution of first and second degree equations. The resolution of these equations was obtained using geometric methods similar to Euclid’s.

Probably inspired by the work of the Indian Brahmagupta, Al-Khawarizmi is also the author of a Treaty of the Indian Number System which introduced the Indian decimal system, especially the zero in his own civilization, but also in Europe, through the Latin translations or adaptations of his works (for example : "Al-Khwarizmi said that..." or the *Book of Fibonacci's Calculs*).

Like many astronomers of that time, Al-Khwarizmi is also an astrologer. According to the historian Tabari, Al-Khwarizmi, together with a group of astrologers, predicted the Caliph's long life (and the fifty years left to live) when he died ten days after the prediction.

2. – Main purpose of the Game.

The purpose of this game is to understand well the formulas by calculating manually the values of a function from the dice roll.

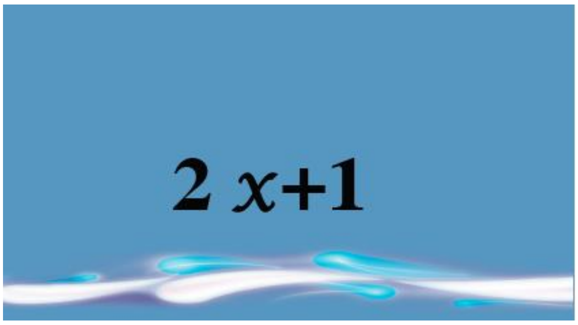
So « HAPPINESS IS IN THE DICE ».

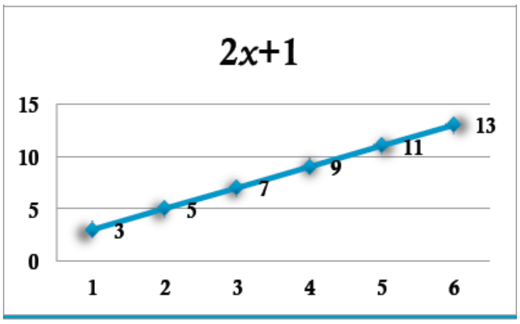
It should be noted that we obviously do not believe that this approach replaces the more structured acquisition of knowledge. But it is a complementary way to encourage people to think and learn.

3. – The material.

On the cards, there are 7 families : family ***x*** in blue, family ***y*** in red, family ***z*** in bright green, family ***t*** in orange, family ***u*** in khaki green, family ***v*** in purple and family ***a*** in yellow.

Each family is composed of 10 cards, each recto card is in color with a formula and has a white verso with the graph of the formula values.





The cards are made by printing the pdf documents. One front and one back pdf are provided for each family of 10 cards. Experience has shown that plasticization is necessary for students’ manipulation. One A4 format per family was used.

There are 7 PowerPoints for teachers' use, one per family. There is no guide, only graphs for video projection. The PowerPoint allows you to discuss the content and ask the students a few questions about the game.

Vocabulary :

Two cards are **true twins** if their different formulas represent the same function.

Two cards are **false twins** if their graphs "look" identical but are not. (Beware of the y-axis scales.)

FAMILIES IN DETAIL

Family *x* : Increasing linear functions. Slope 1 or 2. The PowerPoint contains the 10 graphs followed by a question « True or false twins ? »

Family *y* : Increasing linear functions. Slope 3 or 4. There is a constant function ! The PowerPoint contains the 10 graphs followed by a question « True or false twins ? »

Family *z* : Increasing, constant and decreasing linear functions. The PowerPoint contains the 10 graphs followed by two questions « True or false twins ? »

Family *t* : Increasing and decreasing linear functions. An increasing function with 4 true twin formulas, a decreasing function with 5 twin formulas and an odd one out. The PowerPoint uses several map graphs to ask questions such as « True or false twins ? »

Family *u* : Increasing and decreasing linear functions. Three growing functions with 3 true twin formulas each and an odd one out. The PowerPoint uses several map graphs to ask the question « True or false twins ? », that is, « is it the same function or not ? ».

Family *v* : A single increasing linear function is represented by 10 twin formulas. The PowerPoint shows map graphs to ask the question « True or false twins ? »

Family *a* is not straight ! It consists of 10 quadratic functions. The PowerPoint shows the 10 graphs of these functions.

4. – Game Rules (some suggestions).

It can be played by 2 or 4 individual players or in teams of two players.
Each player or team has one dice (or only one dice for all players but it is slower).
Invention of new rules of the game by the students or adaptation of known rules from other card games is also an important activity. Some examples will be presented in the following section.

SIMPLE GAME WITH A FAMILY

We choose a family, shuffle the cards, put them on the table thanks to the coloured side of the formulas visible.

First we draw the card on the pile.

Each player (or team) rolls his dice.

Each player (or team) calculates the value of the card formula with the number he has drawn with his dice.

We check if each calculation is correct by looking at the back of the card.

Example : card $2(x + 1)$, if dice = 5 then points = 12.

If the calculation is correct, we write down the number of points won by the player, otherwise we give him the smallest winning on this card.

This repressive rule is optional ! We have to find something better. The important thing is that everyone tries to do their calculations. We will see that in practice the students prefer a self-help strategy at the beginning in the calculation, more efficient to progress in the game !

Everyone writes down their points.

We continue one card after another until we finish the family.

The winner (or winning team) is the one who has the highest points at the end of the game.

SIMPLE GAME WITH SEVERAL FAMILIES

We shuffle the cards of several families and we play like with a family.

BATTLE GAME

This game can be played with one or more families. We shuffle the cards. The cards are dealt to the players. To begin with, each player makes his pack of cards without ordering them and without looking at them. The first player puts the first card on the table, « the formula side » in sight, the second does the same. Each player rolls the dice and calculates his points. We check with the graph. The one with the highest points takes both cards. We say "battle" when both players have the same points. In this case, each player takes another card. The winner of a battle game is the one who have has all the cards in his hands.

5. – Rules of the Game (student suggestions).

MATHEMATICAL TOTEM GAME (Lucie®)

This game is played with a minimum of 2 players (team possible).

A dice.

As many families as we want.

A Totem

One of the players, in turn, rolls a dice and all players must calculate the result corresponding to the first card on the pile.

The first to take the Totem while **announcing the result** wins the card if he has given the right result. He puts the card in his "won cards".

If he gives a bad result he puts one of his "won cards" back into the pile plus the played card.

If two players catch the Totem and give the right result at the same time, the one with the largest number of fingers on the Totem takes the card. If they have the same number of fingers, it is the one with the lowest hand.

When all the cards in pile have been played, each player/team counts his or her won cards, the one with the greatest number of cards wins.

Battle Challenges (Miramas® College – *from page 6*)

For groups who have quickly completed all three parts, the teacher invites students to pursue the idea of changing the rules or inventing new ones and thus proceed to a new part. On this occasion, a group chooses to play a "battle challenge" : it is the opposing team that chooses the card, i.e : "When team 1 must roll the die, team 2 chooses the card with which team 1 counts its points. Then when it is up to Team 2 to roll the dice, Team 1 chooses the card, etc. »

With such a rule, "the challenges are greater", students no longer help each other and they use strategies : if the counting student has difficulties, they give him/her a "hard card" in the hope that he/she is wrong and reports 0 points to his/her team. Otherwise they give cards that score as few points as possible. Some students go so far as to try to anticipate the cards. This improvement in the game rule is being successfully tested by the other groups.

Other variants suggested by the students will be described in Chapter 8.

6. – Some possibilities of additional activities.

Search for true twins

Use families ***t*** and ***u*** : to look for true twins and an odd one out.

Graphs with 2 dice

With 2 dice the values of the variables range from 1 to 12. Use checkered paper, a ruler and a pencil. Take a card (family x to start). Students must draw on the sheet of paper the graph for x from 1 to 12. Roll the 2 dice and calculate the points. The same applies to other families. There are surprises for decreasing functions....

Real variable

A magic dice can randomly provide any number between 1 and 6, but also 4.3 or 5.222... or π ! Calculate the number of points for such numbers. Find the results in the graphs. How to make a magic (approximate) dice for real numbers ?

Equation and graphs (inverse problem)

How to solve an equation graphically " in Al-Khwarimi style" ? Take a card. For example in family ***x***, card $2(x\text{-}1)$.

Question : Which x gives 8 points ?

Question + : Which x would give 5 points ?

Question + + + : Which x would give 3.5 points ?

How to solve the problem using a graph ?

Equations and graphics

Take two cards from the same family. Find the x that gives the same number of points in both cards. Does it always exist ?

7. – Fragments of the track logbook.

EXPERIMENTATIONS (school year 2018-2019) :

Sessions of Thursday, March 7 : 1st experiment – 3ème B then 3ème A
(In France, students who are in « 3ème » are 14-15 years old. It is approximately equivalent to Year 10 in the UK and 9th Grade in the US.)

Two groups of 14 or 15 students : the first from 1.30 pm to 2.25 pm and the second from 2.30 pm to 3.25 pm. For each of these two groups the session was organized according to the following protocol :

1st phase : 10 cards of the "x" family are dealt and one dice is in a plastic bag. No calculator.

The rules of the battle game are presented. Students grouped by 4 (two teams of 2 players) are invited to play two or three games ; the teams will then be mixed for a second round !

The students play. They take the time to calculate, check and analyze their classmates methods. We also notice very quickly that students help each other in some groups : all students explain one another even in opposing teams. While some students make it a point of honour to calculate everything from the top, students in difficulty use paper to do their calculations.

This is what the students noticed :

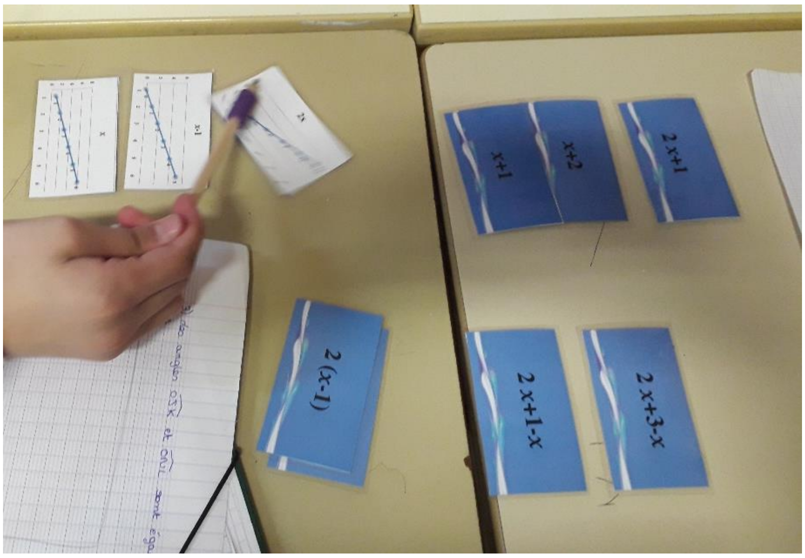
« Some cards pay more than others. » « The checking of results through the verso-graphs is done naturally and without difficulty by all students. » « The idea of using a dice is very good ! » a student judges « because we can catch up if we find a card that gives us a lot of points and make 6. » « Luck plays a role. »

[Battle Challenges (Miramas® College – *See page 5*)]

For groups who have quickly completed all three parts, the teacher invites students to pursue the idea of changing the rules or inventing new ones and thus proceed to a new part. At this occasion, a group chooses to play a « battle challenge » : it is the opposing team that chooses the card, i.e. : « When team 1 must roll the dice, team 2 chooses the card with which team 1 counts its points. Then when it is up to Team 2 to roll the dice, Team 1 chooses the card, etc. »

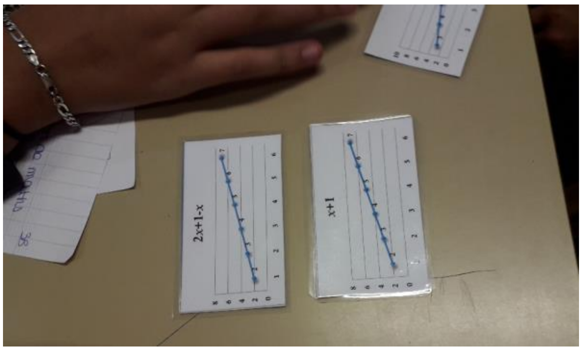
With such a rule, « the challenges are greater », students no longer help each other and they use strategies : if the counting student has difficulties, he or she is given a « hard card » in the hope that he or she is wrong and reports 0 points to the team. Otherwise, he is given cards that win as few points as possible. Some students go so far as to try to anticipate the cards. This improvement in the game rule is being successfully tested by the other groups.

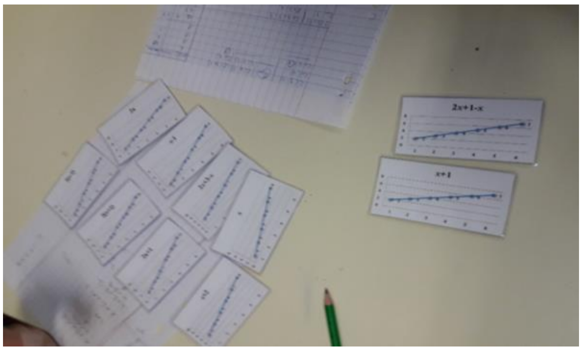
2nd Phase : After several games, the teacher suggests the students to analyze the cards and classify them.



First step : the students focus on the points and more particularly on the maximum they can reach, then they notice that some cards go from 1 in 1, others from 2 in 2 and bring them together according to this criterion. Others note that at the y-axis level, many cards are different.

In a second step, the teacher starts the discussions again and helps some of the students who are analysing the back of the cards. « What do you think of that ? » Students naturally group cards together, but when they return to the front, the writing attracts more their attention than the graphics at first : « there are the $2x$, the x, the $x + 1$, ... ». But then the students go back and forth with the graphs and writings, the students debate and are divided, « what is the most important ? », « the evolution of the results ? » and finally come-back on the « progression of the results » : « here we go from 2 in 2, here from 1 in 1... Yes but there the results are not the same... It's because there is $2x$. » Groups will find that two cards give the same values. Some will explain orally why : « It is enough to reduce the expression. » Others will not look for any explanation.





To conclude : Against all odds, the students remain « cautious » and analyse the maps, calculating slowly « to see how it works ». « These are literal expressions, it's hard », this restraint may come from some mistrust. However, after about ten minutes, once the students have been used to the game, they get involved without any restraint in the game. Then, students change the rules and engage in the « challenge battle ». The desire to modify or optimize the rules is also present. This is a point we should work on in the future. The analysis of the cards shows that algebraic expression is very important to the students. But the graphic aspect attracts students and contributes greatly to the study of cards.

Sessions of Thursday, March 14 : 2nd experiment – 3$^{\text{ème}}$ A then 3$^{\text{ème}}$ B

Two full classes of 24 students : the first from 1.30 pm to 2.25 pm and the second from 2.30 pm to 3.25 pm.

This session follows the first experiment, the students who tested the cards the previous week allow the session to start well. For each class, the session was organized according to the following protocol :

Students are in groups of 4 or 3. It is possible to play successively the following games :

First game (with blue cards) : the « classic » battle. A team rolls the dice and announces the result of the card above the pile. The result is checked by both teams. If the team has announced the right result then it wins the announced points. Otherwise, it marks zero. Each team plays five times. The winning team is the one with the highest total points. This phase is quite similar to the previous session, students take the time to calculate mentally, some do not hesitate to give themselves a "helping hand". But quite quickly all the students make it a point of honour to calculate by themselves.

Second game (with red cards) : the « challenges » battle. (Returning to the idea developed the previous week) a team rolls the dice, the opposing team offers a challenge card. The team who plays must give the right result. If so, it wins the points, otherwise it scores no points. It should be noted that the « 5 » card catches students' attention. Some groups will test the modified rule according which the card is submitted to the opposing team before the dice is rolled. Strategies are then more elaborate on the number of points potentially at stake. Especially since the red cards offer several possibilities. The teams then count point-by-point and one turn after the other to try to anticipate and remain the winner.

The teacher keeps handly the fact that the second game can be played with a third family of cards (the other « colours » are ready to be used) or by mixing families (blue and red for example).

Third game (with a family chosen by the teacher according to the groups), the totem (a glue stick does the trick) : the dice is rolled alternately by each team. The first student who has found the result must grab the totem pole and announce the result at the same time. If the result is correct, the student wins the card for his team. Otherwise the card goes to the opposing team. The team with the greatest number of cards wins.

When there are three students, two students play against each other. The third is the master of the game, he is the one who rolls the dice and checks the announced results. After ten cards have been played, the students switch roles.

While playing, students notice some characteristics about families : « In red cards as in blue cards, there are two cards where there is the same result », « in the green family, it is worse, there are several cards that give the same results. » A student in difficulty who found the trick won several games against other students by playing only with the greens. We can add that what challenges students are the "constant cards" and the cards « where you have to make the smallest possible dice to get the biggest result. » These analyses are not detailed during this second session, they will have to be reinvested during the third experiment. At the end of this session, the teacher suggests the following work at home : the students must improve or, even better, invent the rules of a new game.

Note : A session was held with 15 students of « 5ème » on Tuesday, March 19, 2019 for one hour.
(In France, students who are in « 5ème » are 12-13 years old. It is approximately equivalent to Year 8 in the UK and 7th Grade in the US.)

The session is built on the same principle as the second session with the 3ème. Only the « ***a*** » family is not used with these students. The only difference is that the students, under the circumstances, fully entered the game, without any restraint or caution. The teacher will not have gone as far as analyzing the maps with the students. Apart from that, during the « game » phase strictly speaking, the overall attitude of the students is the same for 3ème and 5ème. Especially when it comes to searching for strategies in game 2.

8. – New suggestions made in writing by the students.

It should be remembered that the students had the choice between improving an existing rule or inventing a new one. Here are the most significant suggestions made :

Tiago : « Teams choose five cards before starting the game. A team rolls the dice and the opposing team offers a card. The team who rolled the dice must calculate the value of the card. If the value is the right one, they win the card otherwise the opposing team keeps the card. At the end of the game (10 dice rolls), the team with the greatest number of cards wins ».

Anthony made several suggestions to make the game tougher.
1) Take the blue cards, proceed to a battle between two teams but knowing that the values of the cards are multiplied by two (The game may evolve if the opposing team proposes (imposes) the multiplier coefficient of its choice.)
2) Take the blue and red card games (place them in parallel). Proceed to a battle but by drawing one card from each game and the results of the two cards will have to be added.
3) Take the red cards, proceed to a battle but knowing that you must automatically subtract (or multiply) the amount of the dice from the result found by the card.

Léa made several suggestions to clarify the rules.
1) It is forbidden to use the same card several times in a game. (Several students will note that this clarification was accepted most of the time by the teams during the experimental sessions).
2) The team can be prohibited from looking at the card which was given by the team who has to calculate the value.
3) When the answer is wrong, subtract the number indicated by the dice from the total points of the team that made the mistake.

Mathis suggests that : « if we make a mistake in the result, we must subtract the right result from our total points, we must count with the relative numbers. Moreover, it's not allowed to look at the result of the card you are going to give to the opponent not to develop too many tactics too easily on points that can be won or lost.

Hugo also made several suggestions to clarify the rules.
1) The team with the advantage must act one player after the other without the help of his teammate...
2) The team who has won a game must necessarily start the next game.

Lenaïc adds that : « If the dice roll number is less than or equal to 3 then the number must be divided by 2 before calculating the result. If the number is equal to or greater than 5 then the number must be multiplied by 2. »

Raphael invents a new version :

Equipment : Take the blue, green and purple cards and one dice.

Rule : Each player in turn rolls the dice. If the number obtained is 2, 3 or 5, the player takes a blue card and must count his points. If the obtained number is 1 or 4, the player takes a green card. Finally, if it is 6, the player takes a purple card. If the player can answer in less than 20 seconds, he wins 1 point or loses 1 point. The first player to win 10 points wins the game.

Remarks : other suggestions are made involving a time limit (10, 20 seconds, ...) or on the number of won points as well as remarks on the contents of the cards « one could add calculations with divisions, fractional writings, ... »

9. – Notes and Bibliography.

This work is the result of a collaboration within the IREM of Aix-Marseille. Direct participants were Olivier Garrigue, Professor of Mathematics at the Collège La Carraire de Miramas (13) and his students, Jorge Rezende, Universidade de Lisboa and Ricardo Lima, Dream & Science Factory and CNRS, Marseille. Thanks to Myriam Quatrini for her encouragement and to Nicole Paoletti and Céline Tallon for their attentive reading.

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Bibliography :

[1] https://ent2d.ac-bordeaux.fr/disciplines/semaine-des-maths/wp-content/uploads/sites/17/2017/03/Al-Khwarizmi-Traduction-Al-Kitāb-al-mokhtasar-fi-hisāb-al-jabr-wa-l-moqābala.pdf

[2] http://eduscol.education.fr/pid34185/cycle-4.html

[3] http://cache.media.eduscol.education.fr/file/programmes_2018/20/4/Cycle_4_programme_consolide_1038204.pdf

[4] http://eduscol.education.fr/cid99696/ressources-maths-cycle-4.html

[5] http://cache.media.eduscol.education.fr/file/Maths_par_le_jeu/92/4/01-RA16_C3_C4_MATH_math_jeu_641924.pdf

[6] http://www.jeuxmath.be

[7] https://www.reseau-canope.fr/