

Institute of Research in Mathematical Education**Aix-Marseille University****About AZULEJOS**

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1. A pedagogical itinerary.

This pedagogical itinerary took place during one session every two or three weeks during the year 2016-2017 with two 6^e (in the French school system, pupils in 6e are about 11/12 years old, which approximately corresponds to Year 7 in the UK and 6th Grade in the US) from the Collège La Carraire of Miramas (Bouches-du-Rhône ; France). It was repeated in 2017-2018 and 2018-2019 with many others 6^e Classes.

During the first session of the year, each group of 3 pupils discovers a collection of 12 plasticized azulejos in an envelope. They have particular geometric properties. After a period of observation and discussion, they start with a series of mathematical questions.

Along the way, the students raised new questions and found answers in various fields such as : symmetries with respect to axes, translation and rotation symmetries and their compositions ; paving stones and their fundamental domains; the enumeration and combinatorics of friezes; the notion of powers and their effectiveness; the divisibility of numbers and their utility; the invention of new forms of azulejos and the recognition of their properties, etc. Although the orientation of the lesson was guided by the dynamics of the pupils' questions and discoveries, the teacher intervened when there was too much disturbing dispersion.

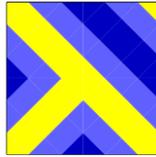
It should be noted that we obviously do not believe that this approach replaces the more structured acquisition of knowledge. But it is a complementary way to encourage people to think and learn.

2. About azulejos.

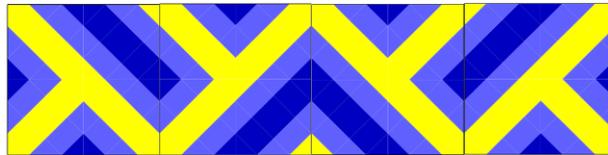


Contumil Station, Porto, Portugal. Azulejos panels, Eduardo Nery, 1993-1994.

Eduardo Nery (1938-2013) is a Portuguese artist who, with others, devoted part of his work to "the aesthetic revaluation of everyday urban spaces...",[1]. This is one of the azulejos created by Eduardo Nery :



It should be noted that these azulejos can be assembled in all possible orientations, respecting a continuity of colours and patterns :



This is only one of the mathematical properties of this pattern. In fact, [2], this pattern and many others invented later [3] have remarkable symmetry properties that combine surprisingly well in the multiple paving that results. These properties use, among other things, the finite group theory [4], [5].

3. What kind of teaching ?

The first steps that guided the development of this activity can be summarized as follows :

An approach : At the beginning of the year, we give pupils a guideline. Despite the hazards that may arise from the school's own functioning, they will be the actors of sessions independent from the usual teaching they daily receive.

An artifact : the plasticized azulejos presented at the beginning as "a treasure" hidden in an envelope ("educational packaging"), will be studied throughout the year. During the sessions, pupils will discover different azulejos and a large number of questions. To create the plasticised azulejos we used pdf files provided by Jorge Rezende.

A logbook : Each pupil has her/his own logbook, it is the main thread of her/his reflection. She or he has access to it throughout the project and writes and draws what she or he believes important to better understand the nature of the studied object.

Two principles : The first is an open approach, i.e. the pupils are at the origine of the progress of the lesson and the reflection as much as possible. The second is the respect the rhythm of each pupil. In 6° pupils are often enthusiastic and so we obviously noted that pupils' reflection could easily drift in all directions. This is where the role of the teacher comes into its own. In this type of teaching, she or he must guarantee the respect of the framework, by sometimes refocusing the pupils' reflection through a selection of strategic questions.

Let us add that there are powerful computer tools available in free access, e.g. [6], which can be used by pupils to build and study paving.

4. What happened ?

We have identified three phases of work that mark this itinerary and which can be also found during all session. The repetition of the same phases are very beneficial in the pupils' work and make their thinking more effective. This sequence is also checked at each session.

1- Adaptation phase :

The pupils discover the working environment and adapt to the situation proposed by the teacher. This is a key moment. As they open the envelope they discover the azulejos and become familiar with this object. We most often notice that pupils ask themselves a lot of questions during this phase on :

- how to arrange tiles to obtain "special shapes" for example. The search for symmetries quickly appeared about the paving but also for patterns that went beyond the azulejos to extend to the friezes and paving.
- the number of possibilities to generate paving stones, friezes, etc.

The teacher must be attentive to this questioning because from it, *strategic questions* can develop and will help pupils to better construct their reflections.

2- Phase of thinking in action :

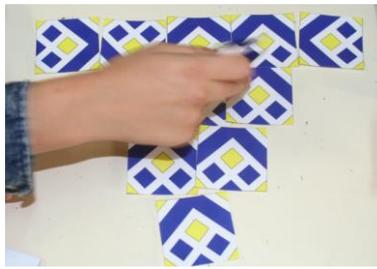
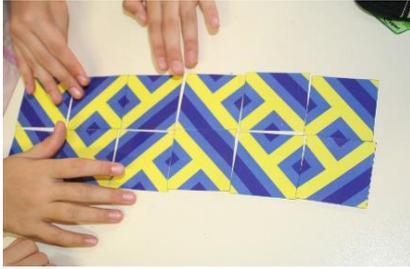
First, and as a transition, this phase may provide an opportunity to work more thoroughly on paving and symmetry as well as on friezes and slipped patterns, but the main objective of this phase remains to support pupils' thinking and to allow them to explicitly formulate their questions and opinions. They continue to manipulate the azulejos to better explain to their classmates and/or the teacher their points of view, their questions. It is important to show pupils that they must be precise and rigorous in their arguments.

3- Development phase of the questioning and reflection :

At the end of the second phase, the teacher should summarize the pupils' reflections. Until then, in our experiments, the question of enumeration is often one of the main questions, as well as how azulejos are designed to have such properties. The teacher can then ask the number of different possibilities to create a frieze of two tiles, then three, etc. Here, the invention of an effective notation system is decisive. Establishing strict rules to build an azulejo also seems to be a good way for pupils to work and may be closely related to the issue of enumeration. However, if we return to the principle of an itinerary, the pupils may not take this direction at all and it will then be up to the teacher to show the pupils new ways.

5. Fragments of the itinerary logbook.

Discovery :



Paving and their symmetry :



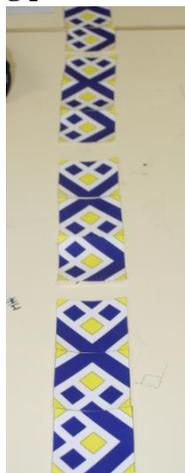
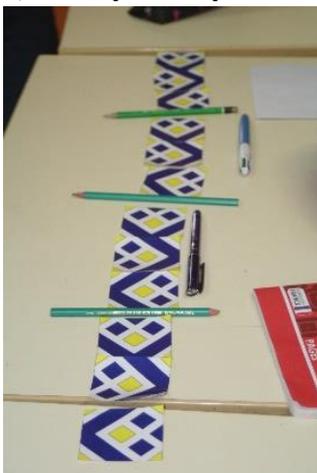
Some groups try...



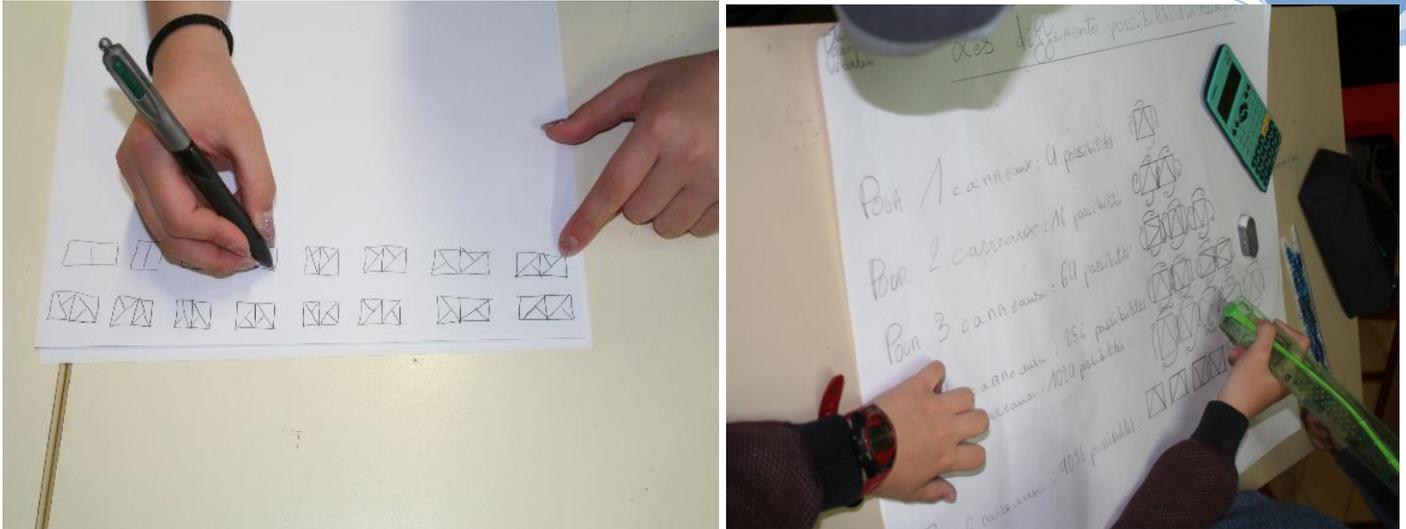
... and fix :



The friezes, axial symmetry and sliding patterns :



Once the possibilities have been counted, the students develop a mathematical notation.



Examples of logbooks :



Here is an example of a moment of synthesis on the board : the enumeration with more than three azulejos :

4 azulejos : $64 \times 4 = \overbrace{4 \times 4 \times 4 \times 4}^{4 \text{ fois}} = 256 \rightarrow 16 \times 16 = 256$

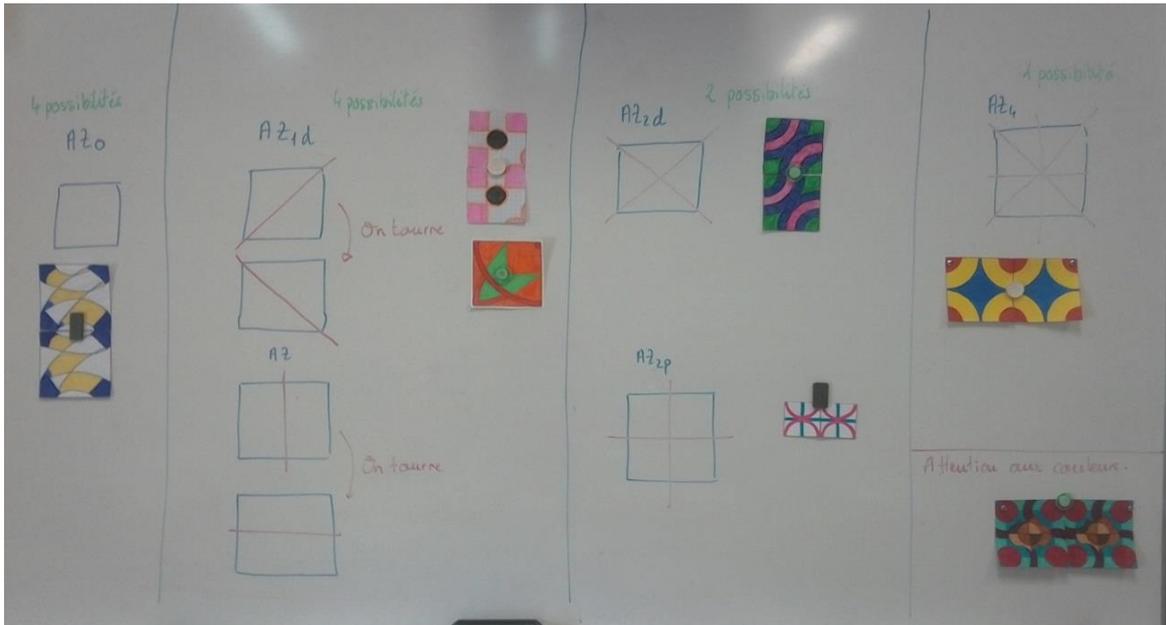
5 azulejos : $256 \times 4 = \overbrace{4 \times 4 \times 4 \times 4 \times 4}^{5 \text{ fois}} = 1024$

6 azulejos : $1024 \times 4 = \overbrace{4 \times 4 \times 4 \times 4 \times 4 \times 4}^{6 \text{ fois}} = 4096$

10 azulejos : $4 \times 4 = 1\,048\,576$ ~~$\triangle 10 \times 4 = 40$~~

12 azulejos } $\underbrace{4 \times 4 \times \dots \times 4}_{12 \text{ fois}} = 16\,777\,216$ possibilités de pavages ou de frises.

After having created azulejos and enumerated all the possibilities in all cases, pupils start a classification :



Here are three excerpts noted during the itinerary, to set the tone.

1. During the first session, 6^e B class.

The pupils are interested in the possibilities offered by the azulejos, some try to obtain small squares « as many as possible » (6th example), others naturally make a 4x3 paving : « 12 it is 4x3, go ahead, make a rectangle of 4 by 3. » « It's like a puzzle. ». Some groups are starting to look for possible assembly variants.

Once each production has been photographed, the teacher goes on to say : « But then how many different paving stones can there be ? »

Many students raise their hands to say : « an infinity ! »

Three or four pupils seem to disagree, one of them raises his hand and asserts : « There can't be an infinite number of possibilities, there are only 12 azulejos. »

Another wants to speak : « Yes, it limits the possibilities. »

The teacher asks : « Why ? »

The first pupil answers : « Well, 12 azulejos is not infinite. »

The technology teacher (who was exceptionally present at this session) suggests : « We can put them as we please, can't we ? »

A pupil replies : « Yes, but in fact each azulejo has only 4 possibilities so it won't go on to infinity. »

The teacher wants to know if he is sure of that.

The pupil argues : « Yes, 12 tiles with 4 possibilities each, that's a lot but it doesn't make it infinite. »

The teacher finally asks : « Do you want to go further with your idea ? »

The pupil just says : « No. ».

2. During the third session, 6^e B class.

Many pupils want to express themselves, the teacher chooses to stay at the board to better manage the debate. Until the end of the session, he or she will record the pupils' suggestions on the whiteboard.

One pupil (in pair number 8) who had not yet expressed herself wants to explain why 64 combinations are required :

- « The reasoning is the same, for 2 azulejos have a $4 \times 4 = 16$ possibilities, so for 4 we have to calculate $4 \times 4 \times 4 \times 4$, so we already know that $4 \times 4 = 16$, we just have to calculate 16×4 . »

- « And that makes 56 ! » his neighbor says.

- « No, that's 64 ! » the interviewed pupil replies.

Fifteen minutes later.

The teacher : « Do you all agree ? »

Pupil #1 : « It is ultimately easier to count the number of possibilities than to record all the combinations... »

The teacher : « I don't know if it's right, Class, what do you think of that ? »

Pupil #2 : « I prefer to count and we can assemble the multiplication by 4. »

The teacher : « What you're saying is very interesting. For example, if I ask you to give me the number of combinations for 6 azulejos, what can you answer me ? » (written in the middle of the blackboard)

Pupil #2 : « For 4 azulejos, I will make $4 \times 4 \times 4 \times 4$ that gives 16×16 and finally the result is 256." »

The teacher : « And you worked it out in your head ? »

Pupil #2 : « No, I took my calculator for 16×16 . »

The teacher : « Okay, but what about six azulejos then ? »

Pupil #3 : « We can do the same as for 3 azulejos, we know that : $4 \times 4 \times 4 \times 4 = 64$ so there I make $4 \times 4 \times 4 \times 4 \times 4 \times 4$ which equals $64 \times 64 = 4096$. But then I also took my calculator for 64×64 . »

Pupil #4 : « Yes, when the calculations are repeated, using the calculator is interesting ... »

The teacher : « Since there is some time left (It is 10.51, 9 minutes left until the end of the lesson) : if I want to calculate: $4 \times 4 \times 4$, how can I do it ? »

Pupil #5 : « If we group the numbers by 2, it gives : $16 \times 16 \times 16 \times 16 \times 16 \times 16$, and after 64×64 ... there is still one left, we still have to write $\dots \times 16$. »

Pupil #6 : « It still takes a long time ! »

Pupil #7 : « But isn't it a story of power ? »

Pupil #8 : « No of square root ! »

The teacher : « What you're saying is very interesting, but do you know what you're talking about ? »

Pupil #9 : « It's not the square root, it's a kind of 'V' like that (the pupil draws the symbol of the square root). »

The teacher on the board : "Like that ? »

Pupil #9 : « Yes like that ! »

The teacher : « So, power or square root ? »

Pupil #7 : « I think it's power. »

The other pupils do not speak out but talk to each other. The lesson is coming to an end.

Teacher : « I ask you to think about that, do your research and we'll talk about it again. »

[End of the lesson]

3. During the fifth session, 6^e B class.

The aim of this session was how to classify the patterns invented by the students into families.

The pupils get to work and talk at the same time with their neighbours, the teacher tolerates the discussions but will ask twice to lower their voice to whisper. He walks in the middle of the pupils to look at what they're doing but he will not intervene with the pupils except to encourage them to draw the possibilities they "imagine".

After 10 minutes, the pupils make the following assessment :

« There are several possibilities to build an azulejo, you can make an azulejo without any symmetry axis (like P6), with 1 symmetry axis, 2 symmetry axes (P1 but also P2 and Odelya's) or 4 symmetry axes. » Four families are thus created and detailed. Notice that the notations "AZ..." are given by the teacher and very quickly adopted by the pupils.

And fifteen minutes later...

Pupil #1 : « In fact, if there are not many axes of symmetry, there are more possibilities. »

The teacher : « Can you be more specific ? How many are there in each case ? »

Pupil #1 : « For 0 or 1 axis, there are 16 possibilities because 4×4 equals 16, for 2 axes there are 4 possibilities and for 4 axes there is only one possibility. »

The teacher : « Well, I cleaned up the board to write down the gist of what you told me, it will be our conclusion, our synthesis on the azulejos, copy it on your research logbooks before it rings. »

[End of the lesson]

6. Notes :

This work is the result of a collaboration within the IREM of Aix-Marseille. Direct participants were Olivier Garrigue, Professor of Mathematics at the Collège La Carraire de Miramas (13) and his students, Jorge Rezende, Universidade de Lisboa and Ricardo Lima, Dream & Science Factory and CNRS, Marseille. Thanks to Myriam Quatrini for her encouragement and to Nicole Paoletti and Céline Tallon for their attentive reading.

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- [5] André Deledicq et Raoul Raba, *Le monde des pavages – Les voir et les faire* – ACL – Les éditions Kangourou. Mars 2014.
- [6] <http://pascal.peter.free.fr/pavages.html> Pavages 2 : a free software to create paving of the plan.